Appendix to
"Neighborhood Dynamics and the Distribution of Opportunity"

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Outline of the Appendix

We begin by presenting a proof of the existence of a general equilibrium in a model with uninsurable idiosyncratic ability risk and a production externality, but no housing sector and no mobility. We then generalize this proof to allow for housing, showing the existence of a Segregated Recursive Competitive Equilibrium (SRCE) from the model without moving used in the paper. The broad outline is as follows:

- Simplified Model without Moving
  - **A.2-A.4** One neighborhood and households get no utility from housing
    - **A.2** We first state the household's problem and use results from Stokey et al. (1989), henceforth SLP, to show that it has a unique solution (i.e., a value function and optimal decision rule) and that the associated value function and decision rule have desirable properties.
    - **A.3** We then show that a unique stationary distribution of human capital and ability (h, a) exists for each parameterization of the model by appealing to Theorem 2 of Hopenhayn and Prescott (1992).
    - **A.4** Because all of these results apply to an economy in which the externality is fixed to be some level χ (and wages are also externally set to some level ω), we conclude by showing that there is an externality χ^* satisfying the required general equilibrium conditions.
- Model without Moving Used in the Text
  - **A.5** We then generalize this proof to the model under segregation used in the text in which households get utility from both consumption and housing

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- Model with Moving Used in the Text

  B.1 We then discuss the conditions that would be required for existence of a general equilibrium of the model with moving used in the text.

A Appendix: Proof of Existence of a SRCE

A.1 Proof: The Household's Problem Has a Unique Solution for Fixed Externality χ

A.1.1 The Household's Problem

We begin by analyzing a model without a moving decision in which the externality χ is not an equilibrium object, but rather is externally set to some fixed value (recall that the wage ω is also partial equilibrium in our model). Because households optimize facing this value, we often use the subscript $_\chi$ to explicitly indicate that sets, decision rules, etc. pertain to the model with the fixed value χ. The state vector of an infinitely-lived household is (h_t, a_t), where the endogenous state variable is human capital $h_t \in \mathbb{H}_\chi \equiv [\underline{h}_\chi, \overline{h}_\chi] \subset R_+$ with $0 < \underline{h}_\chi < \overline{h}_\chi < \infty$, and the exogenous shock is ability $a_t \in \mathbb{A} \equiv \{\underline{a} = a_1, a_2, \ldots, a_k = \overline{a}\} \subset R_+$, which is a stationary Markov chain with transition probabilities denoted by $\pi(a_i|a_j)$ (We assume $a_i < a_{i+1}$ for $i = 1, \ldots, k-1$.). The correspondence $\Gamma_\chi : \mathbb{H}_\chi \times \mathbb{A} \rightrightarrows \mathbb{H}_\chi$ describes the set of all feasible actions taken by the household at time t, whose image is $\Gamma_\chi(h_t, a_t) \subset \mathbb{H}_\chi$. We denote the graph of Γ_χ by

$$\mathrm{gr}\Gamma_\chi \equiv \{(h_t, a_t, h_{t+1}) \in \mathbb{H}_\chi \times \mathbb{A} \times \mathbb{H}_\chi : h_{t+1} \in \Gamma_\chi(h_t, a_t)\},$$

and at times we will also refer to the feasibility correspondence for a fixed $a_i \in \mathbb{A}$, Γ_{χ,a_i} and its graph $\mathrm{gr}\Gamma_{\chi,a_i} \equiv \{(h, h') \in \mathbb{H}_\chi \times \mathbb{H}_\chi : h' \in \Gamma_\chi(h, a_i)\}$. Letting $\mathcal{H}_\chi$ be an element of the Borel $\sigma-$algebra over $\mathbb{H}_\chi$ (denoted $\mathcal{B}(\mathbb{H}_\chi)$) and $\mathcal{A}$ be an element of the finite $\sigma-$algebra generated by the singletons $\{a_i\}_{i=1}^k$ (denoted $\sigma(\{a_i\})$), let $x \in S = \mathbb{H}_\chi \times \mathbb{A}$ and define $\mathcal{S}_\chi$ to be the product $\sigma-$algebra generated by $\mathcal{B}(\mathbb{H}_\chi)$ and $\sigma(\{a_i\})$, containing subsets of the form $B = \mathcal{H}_\chi \times \mathcal{A}$.

The household's preferences over streams of consumption $\{c_t\}_{t=0}^\infty$ are described by the discounted expected sum of period utility $u(c_t)$:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t).$$

Each period a household chooses consumption c_t and next period's human capital h_{t+1} while respecting the period budget constraint

$$c_t + F(a_t, \chi_t, h_{t+1}) \leq \omega_t h_t$$

where ω_t is the city-wide wage and $F : \mathbb{A} \times \mathbb{H}_\chi \times \mathbb{H}_\chi \to \mathbb{R}$ is the cost of producing h_{t+1} units of human capital given this period's ability a_t and neighborhood externality χ_t. If the budget constraint holds with equality, the feasible set of consumption given χ and a_i is

$$\mathbb{C}_{\chi,a_i} \equiv \{c \in \mathbb{R} | c(h,h') = \omega h - F(a_i,\chi,h') \text{ with } (h,h') \in \mathrm{gr}\Gamma_{\chi,a_i}\}.$$

We assume the cost function $F(a_t, \chi_t, h_{t+1})$ is twice continuously differentiable. In Section A.1.2 we state assumptions about the derivatives of F that will be crucial in our proofs.

We can formulate the household's problem recursively as:

$$V_\chi(h,a_i) = \max_{c,h'} \; u(c) \; + \; \beta \sum_{j=1}^{n} V_\chi(h',a_j)\, \pi(a_j|a_i) \tag{1}$$

$$\text{s.t.} \quad c + F\left(a_i,\chi,h'\right) \le \omega h \tag{2}$$

because the following conditions hold for our model:

Condition 1: $\Gamma_{\chi,a_i} : \mathbb{H}_\chi \rightrightarrows \mathbb{H}_\chi$ is non-empty, compact-valued, and continuous for all $a_i \in \mathbb{A}$;
Condition 2: $u : \mathbb{C}_{\chi,a_i} \to \mathbb{R}$ defined by $u[c(h,h')]$ is bounded and continuous;[1]
Condition 3: $u : \mathbb{H}_\chi \to \mathbb{R}$ defined by $u[c(\cdot,h')]$ is strictly increasing;
Condition 4: $\Gamma_{\chi,a_i} : \mathbb{H}_\chi \rightrightarrows \mathbb{H}_\chi$ is increasing for all $a_i \in \mathbb{A}$;
Condition 5: For all $a_i \in \mathbb{A}$, $\theta \in (0,1)$, and pairs (h_1,h_1'), $(h_2,h_2') \in \mathrm{gr}\Gamma_{\chi,a_i}$,
$u : \mathbb{C}_{\chi,a_i} \to \mathbb{R}$ satisfies

$$u\left[\, c\Big(\theta(h_1,h_1') + (1-\theta)(h_2,h_2')\Big) \,\right] \ge \theta u\left[\, c(h_1,h_1') \,\right] + (1-\theta) u\left[\, c(h_2,h_2') \,\right];$$

Condition 6: $\mathrm{gr}\Gamma_{\chi,a_i}$ is convex for all $a_i \in \mathbb{A}$.

More specifically, Conditions 1 and 2 ensure there exists a unique value function $V_\chi : \mathbb{H}_\chi \times \mathbb{A} \to \mathbb{R}$ solving Equation 1 with a non-empty set of feasible plans (Thm 9.6 of SLP). Furthermore, the recursive formulation of the problem has desirable properties. Conditions 3 and 4 ensure that $V_\chi(\cdot,a) : \mathbb{H}_\chi \to \mathbb{R}$ is strictly increasing (Thm 9.7 of SLP), while Conditions 5 and 6 ensure that $V_\chi(\cdot,a)$ is strictly concave and that the optimal policy rule (ie, the decision rule $g_{h,\chi} : \mathbb{H}_\chi \times \mathbb{A} \to \mathbb{H}_\chi$) is a continuous (single-valued) function (SLP Thm 9.8), and therefore also measurable.

A.1.2 Proofs that the Model Satisfies Conditions 1-6

Throughout our proofs we use assumptions about the first derivatives of F:

$F_1 \le 0$: A higher child's ability does not increase the cost of producing h' units for tomorrow;

$F_2 \le 0$: A higher externality does not increase the cost of producing h' units for tomorrow;

[1]Note that $\mathbb{C}_{\chi,a_i}$ is closed and bounded since $\mathbb{C}_{\chi,a_i} = [\omega \underline{h}_\chi - F(a_i,\chi,h^{max}(\underline{h}_\chi,a_i)),\ \omega\overline{h}_\chi - F(a_i,\chi,\underline{h}_\chi)] = [\underline{c}_{\chi,a_i}, \overline{c}_{\chi,a_i}]$. The return function $F : A \to \mathbb{R}$ in SLP is in our model $u : \mathbb{C}_{\chi,a_i} \to \mathbb{R}$. We define the function from SLP $F(\cdot,y,z) : A_{yz} \to \mathbb{R}$ as $u[c(\cdot,h';a_i)] : \mathbb{H}_\chi \to \mathbb{R}$.

$F_3 > 0$**:** The marginal cost of producing h' is positive;

as well as maintained assumptions about the second derivatives, cross-derivatives, and combinations of derivatives:

$F_{11}, F_{22} \geq 0$**:** There is a (weakly) diminishing marginal return from ability and the externality;

$F_{33} > 0$**:** There is a (strictly) increasing marginal cost of producing h', ensures $\mathbb{H}_\chi$ is bounded;

$F_{12} \geq 0$**:** The cross-effect of marginal benefits is (weakly) diminishing;

F_{13}**,**$F_{23} \leq 0$**:** The marginal cost is (weakly) falling in current child's ability and the externality.

$F_3 > -F_2$**:** Ensures $\mathbb{H}$ is globally bounded.

$F_{22} + F_{33} > -2\,(F_{23})$**:** Ensures $\mathbb{H}$ is globally bounded.

We also make use of the fact that we can define $\Gamma_{\chi,a_i} : \mathbb{H}_\chi \rightrightarrows \mathbb{H}_\chi$, the correspondence describing the feasibility constraints for a fixed a_i, by:

$$\Gamma_{\chi,a_i}(h) = \left\{ \ h' \in \mathbb{H}_\chi \ \mid \ h' \in [\underline{h}_\chi, h_\chi^{max}(h, a_i)] \ \right\}. \tag{3}$$

Establishing the representation in Equation 3 requires showing that for any χ there exists a maximum h' denoted by $h_\chi^{max}(h, a_i)$ in the choice set of agents with state vector (h, a_i), and that agents can choose any $h' \leq h_\chi^{max}(h, a_i)$ but cannot choose any $h' > h_\chi^{max}(h, a_i)$. This follows from the strict convexity of F in h' ($F_{33} > 0$), as there exists a unique $h_\chi^{max}(h, a_i)$ such that

$$\begin{aligned}
\omega h &> F\Big(a_i, \ \ \chi, \ \ h_\chi^{max}(h, a_i)\Big) && \forall \ \ h' < h_\chi^{max}(h, a_i); \\
\omega h &= F\Big(a_i, \ \ \chi, \ \ h_\chi^{max}(h, a_i)\Big) && \text{if } h' = h_\chi^{max}(h, a_i); \text{ and} \\
\omega h &< F\Big(a_i, \ \ \chi, \ \ h_\chi^{max}(h, a_i)\Big) && \forall \ \ h' > h_\chi^{max}(h, a_i).
\end{aligned}$$

Thus the feasibility correspondence is defined as in Equation 3.

We now show that **Conditions 1, 4, and 6 hold in our model** by showing that the feasibility correspondence has the following properties:

Γ_{χ,a_i} **Is Non-empty:** Γ_{χ,a_i} is clearly non-empty as long as we choose $\underline{h}$ such that $0 < \underline{h} \leq h_\chi^{max}(\underline{h}, a_i)$. We can be assured that $0 < \underline{h} \leq h_\chi^{max}(\underline{h}, a_i)$ for all $a_i \in \mathbb{A}$ as long as $F\Big(\underline{a}, \ \chi \ , \underline{h}\Big) \leq 0$.

Γ_{χ,a_i} **Is Compact Valued:** Γ_{χ,a_i} is compact valued because we can see from Equation 3 that $\Gamma_{\chi,a_i}(h) \subset R_+$ is a closed and bounded, and therefore compact, set for each (h, a_i).

Γ_{χ,a_i} **Is lower hemi-continuous:** We follow the proof of Exercise 3.13 (b) in Irigoyen et al. (2003). Looking at the budget constraint in Equation 2, we can see that $h_{\chi,\omega}^{max}(h, a_i)$ is continuous in h by our assumptions on F. So let $y \in \Gamma_\chi(h, a_i)$. Given a sequence $\{h_m\}_{m=1}^{\infty}$ with $\lim_{m\to\infty} h_m = h$, let $\gamma = y/h_\chi^{max}(h, a_i)$ and define $y_m = \gamma h_\chi^{max}(h_m, a_i)$. Then $y_m \in \Gamma_\chi(h_m, a_i)$ for all m, and since h_χ^{max} is continuous in h, we have that

$$\lim_{m\to\infty} y_m = \gamma \lim_{m\to\infty} h_\chi^{max}(h_m, a_i) = \gamma h_\chi^{max}(h, a_i) = y.$$

Γ_{χ,a_i} **Is upper hemi-continuous:** Consider a sequence $\{(h_m, a_i)\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty}(h_m, a_i) = (h, a_i)$. If there is a sequence $\{h'_m\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} h'_m = h'$ and $h'_m \in \Gamma_{\chi,\omega}(h_m, a_i)$ for all m, then $h'_m \in [\underline{h}, h_\chi^{max}(h_m, a_i)]$ for all m. By the continuity of h_χ^{max}, this implies that

$$\begin{aligned}
&\lim_{m\to\infty} h'_m = h' \geq \underline{h} = \lim_{m\to\infty} \underline{h} \qquad \text{and} \\
&\lim_{m\to\infty} h'_m = h' \leq h_\chi^{max}(h, a_i) = \lim_{m\to\infty} h_\chi^{max}(h_m, a_i),
\end{aligned}$$

so

$$h' \in [\underline{h}, h_\chi^{max}(h, a_i)],$$

proving that $h' \in \Gamma_{\chi,a_i}(h)$.

grΓ_{χ,a_i} **Is Convex:** Consider any two points $h_1, h_2 \in \mathbb{H}_\chi$, and suppose that $h'_1 \in \Gamma_{\chi,a_i}(h_1)$ and $h'_2 \in \Gamma_{\chi,a_i}(h_2)$. As long as $F(a, \chi, h')$ is convex in h' as assumed and the budget constraint is defined as in Equation 2, we know that $h_\chi^{max}(h, a_i)$ is concave. That is, as long as $F(a_i, \chi, h')$ is convex in h',

$$\theta h'_1 + (1-\theta)h'_2 \quad \in \quad \Bigg[\ \underline{h}_\chi\ ,\ \ h_\chi^{max}\Big(\theta h_1 + (1-\theta)h_2,\ \ a_i\ \Big)\Bigg],$$

so the set $\mathrm{gr}\Gamma_{\chi,a_i}$ is convex.

Γ_{χ,a_i} **Is Increasing:** Because $h_1 \leq h_2$ implies $\omega h_1 \leq \omega h_2$ and F is increasing in h', we know that $h_1 \leq h_2$ implies $h_\chi^{max}(h_1, a_i) \leq h_\chi^{max}(h_2, a_i)$. Thus $\Gamma_\chi(h_1, a_i) \subseteq \Gamma_\chi(h_2, a_i)$.

Condition 2 holds in our model: Given χ and a_i, a given value of (h, h') determines consumption when the budget constraint holds with equality, $c : \mathbb{H}_\chi \times \mathbb{H}_\chi \to \mathbb{C}_{\chi,a_i}$ defined by $c(h, h') = \omega h - F(a_i, \chi, h')$. Since c is continuous and $u : \mathbb{C}_{\chi,a_i} \to \mathbb{R}$ is continuous, $u[c(\cdot,\cdot)] : \mathrm{gr}\Gamma_{\chi,a_i} \to \mathbb{R}$ is also continuous. $u[c(\cdot,\cdot)]$ is bounded since $\mathrm{gr}\Gamma_{\chi,a_i}$ is compact valued. **Condition 3 holds in our model** since $u[c(\cdot, h')] : \mathbb{H}_\chi \to \mathbb{R}$ is strictly increasing. And **Condition 5 holds in our model** since given any $(h_1, h'_1), (h_2, h'_2) \in \mathrm{gr}\Gamma_{\chi,a_i}$,

$$\begin{aligned}
&u\Big\{c\Big[\theta(h_1, h'_1) + (1-\theta)(h_2, h'_2)\Big]\Big\} \\
&\qquad = u\Big\{\omega\Big[\theta h_1 + (1-\theta)h_2\Big] - F\Big(a_i, \chi, \theta h'_1 + (1-\theta)h'_2\Big)\Big\} \\
&\qquad > u\Big\{\omega\Big[\theta h_1 + (1-\theta)h_2\Big] - \Big[\theta F(a_i, \chi, h'_1) + (1-\theta)F(a_i, \chi, h'_2)\Big]\Big\} \qquad (4) \\
&\qquad = u\Big\{\theta c(h_1, h'_1) + (1-\theta)c(h_2, h'_2)\Big\} \\
&\qquad > \theta u\Big\{c(h_1, h'_1)\Big\} + (1-\theta)u\Big\{c(h_2, h'_2)\Big\} \qquad (5)
\end{aligned}$$

where 4 follows from the strict convexity of F in h' and 5 follows from the strict concavity of u.

A.2 Proof: A Unique Stationary (h, a) Distribution Exists for Fixed χ

Next, we must show that a unique stationary equilibrium exists for a fixed value of χ. That is, we now prove that there exists a unique distribution $\mu_\chi^* : \mathcal{S}_\chi \to [0, 1]$ perpetually reproducing itself when the optimal decision rule is followed by agents in the economy facing the given ability shock process and an imposed/fixed externality χ and fixed wage ω.

Given the measurable decision rule $g_{h,\chi} : \mathbb{H}_\chi \times \mathbb{A} \to \mathbb{H}_\chi$ and the Markov chain ability process alternatively written using the function $Q : \mathbb{A} \times \sigma\big(\{a_i\}\big) \to [0, 1]$ defined by

$$Q\left(a_i, \mathcal{A}\right) \equiv \sum_{a_j \in \mathcal{A}} \pi(a_j | a_i),$$

we know from Theorem 9.13 in SLP that we can write a well-defined transition function P_χ : $S \times \mathcal{S}_\chi \to [0, 1]$ for the Markov process induced by the household's problem as:

$$P_\chi\left(x, B\right) = P_\chi\left((h, a_i), \mathcal{H}_\chi \times \mathcal{A}\right) = \mathbf{1}\left\{g_{h,\chi}\left(h, a_i\right) \in \mathcal{H}_\chi\right\} Q\left(a_i, \mathcal{A}\right), \tag{6}$$

where $\mathbf{1}\left\{\cdot\right\}$ is the indicator function. The Markov process described by P_χ induces a mapping

$$\Psi_\chi : \mathcal{P}(\mathcal{S}_\chi) \to \mathcal{P}(\mathcal{S}_\chi)$$

from the set of probability measures $\mathcal{P}(\mathcal{S}_\chi)$ into itself that updates probability measures as follows:

$$\mu_{i+1,\chi}(B) = \Psi_\chi(\mu_{i,\chi}(B)) = \int P_\chi\left(x, B\right) \mu_{i,\chi}(dx)$$

Because the following three conditions hold in our model,

Condition I: P_χ is increasing;

Condition II: $\mathbb{H}_\chi \times \mathbb{A}$ has a lower bound $(\underline{h}_\chi, \underline{a})$ and an upper bound $(\overline{h}_\chi, \overline{a})$; and

Condition III: There exists $(h_\chi^*, a^*) \in \mathbb{H}_\chi \times \mathbb{A}$ and a natural number m such that $P_\chi^m\big((\underline{h}_\chi, \underline{a}),\ [\underline{h}_\chi, h_\chi^*] \times \{\underline{a}, \ldots, a^*\}\big) > 0$ and $P_\chi^m\big((\overline{h}_\chi, \overline{a}),\ [h_\chi^*, \overline{h}_\chi] \times \{a^*, \ldots, \overline{a}\}\big) > 0$;

we can appeal to Theorem 2 in Hopenhayn and Prescott (1992) as a proof that for each χ there exists a unique stationary distribution μ_χ^* such that

$$\mu_\chi^*(B) = \Psi_\chi(\mu_\chi^*(B)) = \int P_\chi\left(x, B\right) \mu_\chi^*(dx).$$

A.2.1 Proof of Condition I: P_χ Is Increasing

We begin by proving that $\boldsymbol{g_{h,\chi}}$ **is strictly increasing in** h**:** Given χ and a_i, let h_1 and $h_2 \in \mathbb{H}_\chi \subset \mathbb{R}_+$ with $h_1 > h_2$. Supposing by way of contradiction that $g_{h,\chi}\left(h, a_i\right)$ is not strictly increasing in h, this would imply:

$$g_{h,\chi}(h_1, a_i) \leq g_{h,\chi}(h_2, a_i). \tag{7}$$

However, because $\omega h_1 > \omega h_2$ and $F\Big(a_i, \chi, g_{h,\chi}(h_1, a_i)\Big) \leq F\Big(a_i, \chi, g_{h,\chi}(h_2, a_i)\Big)$ (by our assumptions that $F_3 > 0$), we know that

$$g_{c,\chi}(h_1, a_i) = \omega h_1 - F\Big(a_i, \chi, g_{h,\chi}(h_1, a_i)\Big) > \omega h_2 - F\Big(a_i, \chi, g_{h,\chi}(h_2, a_i)\Big) = g_{c,\chi}(h_2, a_i).$$

This implies $u_c\big[c(h_1)\big] < u_c\big[c(h_2)\big]$, which again by our assumptions on F ($F_{33} > 0$) implies

$$u_c\big[g_{c,\chi}(h_1, a_i)\big] F_3\Big(a_i, \chi, g_{h,\chi}(h_1, a_i)\Big) < u_c\big[g_{c,\chi}(h_2, a_i)\big] F_3\Big(a_i, \chi, g_{h,\chi}(h_2, a_i)\Big).$$

By the Euler Equation this tells us that

$$V_h\big(g_{h,\chi}(h_1, a_i), a'\big) < V_h\big(g_{h,\chi}(h_2, a_i), a'\big),$$

implying that $g_{h,\chi}(h_1, a_i) > g_{h,\chi}(h_2, a_i)$ by the strict concavity of V, contradicting Inequality 7.

We next show that $\boldsymbol{g_{h,\chi}}$ **is strictly concave in** $\boldsymbol{h}$**:** The optimal investment function $g_{i,\chi} : \mathbb{H}_\chi \times \mathbb{A} \to \mathbb{R}$ and human capital decision rule $g_{h,\chi} : \mathbb{H}_\chi \times \mathbb{A} \to \mathbb{H}_\chi$ imply an optimal investment cost function that can be defined as

$$i^*_\chi(h, a) = F(a, \chi, g_{h,\chi}(h, a))$$

Fixing both χ and ability a_j, we can write the production cost function as a function of this period's human capital alone, h. We denote this function as

$$F_{\chi;a_j} : \mathbb{H}_\chi \to \mathbb{R},$$

which further allows us to write $i^*_{\chi;a_j} : \mathbb{H}_\chi \to \mathbb{R}$ defined by the rule

$$i^*_{\chi;a_j}(h) = F_{\chi;a_j}(a_j, \chi, g_{h,\chi}(h, a_j)).$$

Thus one can define $g_{h,\chi;a_j} : \mathbb{H}_\chi \to \mathbb{H}_\chi$

$$g_{h,\chi;a_j}(h) = F^{-1}_{\chi;a_j}(i^*_{\chi;a_j}(h))$$

Since F is strictly convex in h', we know that $F^{-1}_{\chi;a_j}$ is strictly concave. To ensure that $g_{h,\chi;a_j}$ is strictly concave in h, therefore, we need only prove that $i^*_{\chi;a_j}$ is strictly increasing.

To see that $i^*_{\chi;a_j}$ is strictly increasing, consider that the Euler Equation can be written as

$$F_3(a_i, \chi, h')u'(c) = \beta E[V_1(h', a')].$$

Recall that the binding budget constraint implies $c + i^*_{\chi;a_j}(h) = \omega h$. Supposing by way of contradiction that $i^*_{\chi;a_j}(h)$ were weakly decreasing in h, this would imply that c were strictly increasing in h. Then $u'(c)$ would decrease in h, requiring that $E[V_1(h', a')]$ would decrease, or that h' were increasing in h. But this provides the contradiction to the supposition that $i^*_{\chi;a_j}(h)$ was weakly decreasing in h, so $i^*_{\chi;a_j}$ must be strictly increasing.

Finally, we show that $\boldsymbol{g_{h,\chi}}$ **is strictly increasing in** a**:** Suppose not. Then there exist a_1, a_2, and h_0 such that $a_1 > a_2$ and $g_{h,\chi}(h_0, a_1) \leq g_{h,\chi}(h_0, a_2)$. Since F_3 is strictly increasing in h' and strictly decreasing in a, this implies that $F(a_1, \chi, g_{h,\chi}(h_0, a_1)) < F(a_2, \chi, g_{h,\chi}(h_0, a_2))$. Since wage income is the same in either case, by the budget constraint it must be that $g_c(h_0, a_1) > g_c(h_0, a_2)$. By the properties of u, $u_c(g_c(h_0, a_1)) < u_c(g_c(h_0, a_2))$. From the Euler equation,

$$EV_h\big(g_{h,\chi}(h_0, a_1), a'\big) F_3(a_2, \chi, g_{h,\chi}(h_0, a_2)) > EV_h\big(g_{h,\chi}(h_2, a_i), a'\big) F_3(a_1, \chi, g_{h,\chi}(h_0, a_1))$$

By the properties of F, $F_3(a_2, \chi, g_{h,\chi}(h_0, a_2)) < F_3(a_1, \chi, g_{h,\chi}(h_0, a_1))$ so

$$EV_h\big(g_{h,\chi}(h_0, a_1), a'\big) < EV_h\big(g_{h,\chi}(h_0, a_2), a'\big)$$

If the ability process is i.i.d so that $E(a'|a_1) = E(a'|a_2)$, then this implies $g_{h,\chi}(h_0, a_1) > g_{h,\chi}(h_0, a_2)$ which is a contradiction. If instead, $E(a'|a_1) > E(a'|a_2)$, then by the Envelope condition

$$V_h\big(h', a'\big) = u_{c'} w'$$

where $w' = w$ in a steady state. Differentiating $V_h\big(h', a'\big)$ with respect to a' yields

$$\frac{u^2}{c'^2}\left(-F_1 w'\right) > 0$$

The sign comes from the properties of u and F, specifically that the second derivative of u is negative and $F_1 \leq 0$. Thus, we return to the same contradiction that $g_{h,\chi}(h_0, a_1) > g_{h,\chi}(h_0, a_2)$.

A.2.2 Proof of Condition II: $\mathbb{H}_\chi$ Is Closed and Bounded

For a given χ, take (h, a_i) as given, and suppose that $\{h_n\}_{n=1}^{\infty} \subseteq \mathbb{H}_\chi$ with $h_n \to h_0$. Then $c + F(a_i, \chi, h_n) \leq \omega h$ for all $n \in \mathbb{N}$, which by the continuity of F implies that $c + F(a_i, \chi, h_0) \leq \omega h$, so that $h_0 \in \mathbb{H}_\chi$. Thus $\mathbb{H}_\chi$ is closed for each χ.

Recall that the cost of endowing h' units of human capital in a child is a function $F : \mathbb{A} \times \mathbb{H}_\chi \times \mathbb{H}_\chi \to \mathbb{R}$ defined by $F(a, \chi, h')$, where $F \geq 0$ and equality holds at $F\left(\underline{a}, \chi, \underline{h}_\chi\right)$. Central to our proof that $\mathbb{H}_\chi$ is bounded is the **lowest maintenance cost function** $f : \mathbb{H}_\chi \times \mathbb{R} \to \mathbb{R}$, whose rule tells us the cost of maintaining human capital level h given the best possible ability shock when facing the (non-equilibrium/externally set) externality χ:

$$f(h, \chi) = F(\overline{a}, \chi, h).$$

As long as f is strictly convex in h, then given any χ there is some $\overline{h}_\chi < \infty$ such that

$$\begin{aligned}
F(\overline{a}, \chi, h) &< \omega h \quad \text{if} \quad h < \overline{h}_\chi; \\
F(\overline{a}, \chi, h) &= \omega h \quad \text{if} \quad h = \overline{h}_\chi; \\
F(\overline{a}, \chi, h) &> \omega h \quad \text{if} \quad h > \overline{h}_\chi,
\end{aligned}$$

so that $[\underline{h}_\chi, \overline{h}_\chi]$ determines the feasible set of h'.

A strictly convex f in h requires that f has a strictly-positive third derivative, or that

$$\frac{\partial f}{\partial h} = F_3 f\,(h) > 0,$$

or that

$$F_3 > 0. \tag{8}$$

That is, as the level of h to be maintained increases, the increase in the marginal cost of maintaining h tomorrow must dominate the increase in the marginal benefits from today's human capital and today's externality.

For f to have a strictly-positive second derivative in h, it must be the case that

$$\begin{aligned}
\frac{d^2 f}{dh^2} &= \left[\frac{\partial}{\partial h}\,(F_{33})\right] f\,(h) + (F_3)\,\frac{\partial f}{\partial h} \\
&= F_{33} f(h) + F_3^2 f(h) \\
&> 0.
\end{aligned} \tag{9}$$

If F is strictly convex, then this condition is clearly satisfied.

A.2.3 Proof of Condition III: The Monotone Mixing Condition (MMC)

Recall that $[\underline{h}_\chi, \overline{h}_\chi] \equiv \mathbb{H}_\chi$ is the interval bounded by the minimum and maximum h attainable for a given χ. Following Huggett (1993), define the sequences $\{y_{\chi,n}\}_{n=1}^{\infty}$ and $\{z_{\chi,n}\}_{n=1}^{\infty}$

$$\begin{array}{llll}
y_{\chi,1} = \underline{h}_\chi & y_{\chi,2} = g_{h,\chi}(y_{\chi,1}, \overline{a}) & y_{\chi,3} = g_{h,\chi}(y_{\chi,2}, \overline{a}) & \cdots \\
z_{\chi,1} = \overline{h}_\chi & z_{\chi,2} = g_{h,\chi}(z_{\chi,1}, \underline{a}) & z_{\chi,3} = g_{h,\chi}(z_{\chi,2}, \underline{a}) & \cdots
\end{array}$$

The fact that $g_{h,\chi}$ is strictly concave in h and strictly increasing in a implies that $y_{\chi,n} \to \overline{h}_\chi$ and $z_{\chi,n} \to \underline{h}_\chi$, as Figure 1 helps to illustrate.

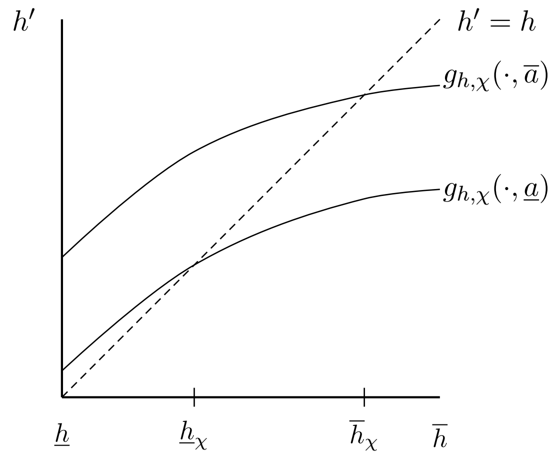


Figure 1: Decision Rules Illustrating Proof from Huggett (1993)

A.3 Proof: A Stationary General Equilibrium Exists

We now show that there exists a general equilibrium, or a human capital level χ^* whose associated steady state equilibrium implied by the model yields an externality of χ^*. Given the global feasible set $\mathbb{H} = [\underline{h}, \overline{h}]$, consider the aggregate supply of human capital that would be implied if χ were externally set to some value. We showed earlier that for any externally fixed χ there is a unique steady state partial equilibrium, which can be characterized by the associated distribution of (h, a), μ^*_χ. Thus we can define the aggregate supply of human capital implied by a given value of χ as:

$$H^S(\chi) = \left\{ H : H = \int g_{h,\chi}(h, a) \; d\mu^*_\chi(h, a) \right\}$$

The equilibrium conditions in the model define a self-map $J : \mathbb{H} \to \mathbb{H}$ by the equation

$$J_1(\chi) = \mathrm{H}(\chi) = \int g_{h,\chi}(h, a) \; d\mu^*_\chi(h, a). \tag{10}$$

The self-map J_1 tells us the externality/level of aggregate human capital implied by the model in which everything is determined in equilibrium except the experienced externality χ, which is set outside the model. A fixed point of J is therefore a general equilibrium of our model.

The Theorem of the Maximum (Thm 4.10.22 in Corbae et al. (2009)) implies $g_\chi(h, a)$ is continuous in χ. Because we also know that

a) $\mathbb{H} \times \mathbb{A}$ is compact;

b) Given a sequence $\{(\chi_n, (h_n, a_n))\}$ that converges to $(\chi_0, (h_0, a_0))$, $\quad P_{\chi_n}((h_n, a_n), \mathcal{H} \times \mathcal{A})$ converges weakly to $P_{\chi_0}((h_0, a_0), \mathcal{H} \times a_0)$ (due to Lebesgue's Dominated Convergence Theorem);

c) For each χ, there is a unique fixed point $\mu^*_\chi : \mathcal{S} \to [0, 1]$ of $\Psi_\chi : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{S})$;

we can appeal to Theorem 12.13 of SLP as a proof that

$$\int g_{h,\chi}\left(h,a\right)\ d\mu_{\chi}^{*}(h,a)$$

is continuous in χ, so $J_1(\chi)$ is continuous as a result.

By the continuity of J_1 and the compactness and convexity of $\mathbb{H}$, we know by Brouwer's Fixed Point Theorem (Aliprantis and Border (2006), Corollary 17.56) that J_1 has a fixed point χ^*, or that our model has a general equilibrium.

To show that J_1 is indeed a self-map on the set $\mathbb{H} = [\underline{h}, \overline{h}] \subset \mathbb{R}_+$, we need to show that the set of feasible household human capital levels, $\mathbb{H}$, is globally closed and bounded. We start by generalizing the proof of $\mathbb{H}_\chi$ being bounded. Consider the specific **lowest maintenance cost function** $f : \mathbb{H} \to \mathbb{R}$ telling us the cost of maintaining human capital level h given the best possible ability shock when facing the externality $\chi = h$,

$$f(h) = F(\overline{a}, \chi = h, h),$$

or assuming the entire population also has the same level of human capital and the highest ability shock (ie, assuming $\mu_\chi\Big((h,\overline{a})\Big) = 1$). As long as $f(h)$ is strictly convex, then we know that there exists a global $\overline{h}$ for which

$$\begin{aligned} F(\overline{a}, \chi = h, h) &< \omega h \quad \text{if} \quad h < \overline{h}; \\ F(\overline{a}, \chi = h, h) &= \omega h \quad \text{if} \quad h = \overline{h}; \\ F(\overline{a}, \chi = h, h) &> \omega h \quad \text{if} \quad h > \overline{h}. \end{aligned}$$

A strictly-positive first derivative of f requires that

$$\frac{df}{dh} = \left[F_2 + F_3\right] f\left(h\right) > 0.$$

This condition is satisfied if and only if

$$F_3 > -F_2. \tag{11}$$

That is, as the level of h to be maintained increases, the increase in the marginal cost must dominate the increase in the marginal benefits from the externality.

Showing f has a strictly-positive second derivative is more tedious. Assuming $\frac{d\chi}{dh} = 1$ and $\frac{d^2\chi}{dh^2} = 0$, we can express the second derivative of f as

$$\begin{aligned} \frac{d^2 f}{dh^2} &= \left[\frac{\partial}{\partial h}\left(F_2 + F_3\right)\right] f\left(h\right) + \left(F_2 + F_3\right)\frac{df}{dh} \\ &= \left[F_{22} + 2F_{23} + F_{33}\right] f\left(h\right) + \left(F_2 + F_3\right)\frac{df}{dh} \\ &> 0. \end{aligned} \tag{12}$$

Now if the term inside of the [] brackets in Equation 12 is positive, then this condition is satisfied. This term will be positive if

$$F_{22} + F_{33} > -2F_{23}. \tag{13}$$

The left hand side terms are disincentives to produce more human capital; the decreased marginal benefit of a higher externality and the increased marginal cost of h'. The right hand side is an incentive to produce more human capital; the decreased marginal cost of producing more h' induced by having higher externality.

A.4 Generalization of Proof to the SRCE Model in the Text (with Housing)

To generalize the proof to a model that includes housing, we first approach the problem with fixed price ρ of housing services $s \in \mathbb{S} \subset \mathbb{R}_+$, thus subscripting by (χ, ρ) rather than χ alone.[2] Some changes to the household's problem are that now the feasible set is defined over a different range $\Gamma_{(\chi,\rho),a_i} : \mathbb{H}_{(\chi,\rho)} \rightrightarrows \mathbb{H}_{(\chi,\rho)} \times \mathbb{S}_{(\chi,\rho)}$, the utility function is defined over a different domain $u : \mathbb{C}_{(\chi,\rho),a_i} \times \mathbb{S}_{(\chi,\rho),a_i} \to \mathbb{R}$, there is a decision rule for housing services $g_{s,(\chi,\rho)}(h,a)$, and the transition function $P_{(\chi,\rho)} : S \times \mathcal{S} \to [0,1]$ for the Markov process induced by the household's problem also depends on ρ:

$$P_{(\chi,\rho)}\left((h,a), \mathcal{H} \times \mathcal{A}\right) = \mathbf{1}\left\{g_{h,(\chi,\rho)}\left(h, a_i\right) \in \mathcal{H}\right\} Q\left(a_i, \mathcal{A}\right). \tag{14}$$

As long as we specify an additive utility function $u(c,s) = \phi(c) + \psi(s)$ with properties analogous to those of $u(c)$ used in the earlier proofs, analogues to the proofs in Sections A.1 and A.2 all hold, up to and including the proof that a stationary distribution $\mu^*_{(\chi,\rho)}$ exists and is unique for fixed χ and ρ. The key insight is that there is an intra-temporal equilibrium condition in this model rendering s as a function of c, so that the arguments using the budget constraint $c + \rho s + F(a, \chi, h') \leq \omega h$ all still hold.

To generalize the proof from Section A.3 to accommodate housing, note that $\mathbb{P}$ is bounded because $\mathbb{H}$ is bounded. To see this, first define the point mass distributions by

$$\begin{aligned} \underline{\mu}\left(\underline{h}, \underline{a}\right) &= 1 \quad \text{and} \\ \overline{\mu}\left(\overline{h}, \overline{a}\right) &= 1, \end{aligned}$$

which allow us to define $\mathbb{P} = [\underline{\rho}, \overline{\rho}]$, where given the associated human capital externalities and

[2] To be clear here, recall that $s \in \mathbb{S}$ is housing services, $x \in S$ is the state vector, and $B \in \mathcal{S}$ is a Borel set in the associated product σ-algebra.

prices $(\underline{\chi}, \underline{\rho})$, $(\overline{\chi}, \overline{\rho})$, we have:

$$\underline{\rho} = \phi\bigg(\int g_{s,(\underline{\chi},\underline{\rho})}\, (h,a) \;\; d\underline{\mu}(h,a) \bigg)$$
$$\overline{\rho} = \phi\bigg(\int g_{s,(\overline{\chi},\overline{\rho})}\, (h,a) \;\; d\overline{\mu}(h,a) \bigg)$$

Recall that we have assumed the absentee landlord supplies housing to meet demand. Thus the equilibrium condition in the housing market is that the price of housing services supports this allocation, which can be used to define $J_2 : \mathbb{H} \times \mathbb{P} \to \mathbb{P}$ by

$$J_2(\chi, \rho) = \rho(\chi, \rho) = \phi\bigg(\int g_{s,(\chi,\rho)}\, (h,a) \;\; d\mu^*_{(\chi,\rho)}(h,a) \bigg). \tag{15}$$

Equation 15 allows us to define the self-map $J : \mathbb{H} \times \mathbb{P} \to \mathbb{H} \times \mathbb{P}$ using J_2 and the analogue to J_1 in Equation 10. We can again apply Brouwer's Fixed Point Theorem to ensure existence of a fixed point (χ^*, ρ^*).

B Appendix: Discussion of the Existence of a MRCE

B.1 Conditions Required to Prove Existence of a MRCE

Here we sketch an outline of a proof of existence of a MRCE (ie, a general equilibrium of the model with two neighborhoods and mobility). The outline does not constitute a proof because there is one condition that we cannot show analytically. Nevertheless, we believe this condition does hold for a region of the parameter space, and the sketch should give the reader some intuition about the model with moving.

To begin, the state vector becomes (h, a, n), where $n \in \{N1, N2\}$ denotes neighborhood. Since we now must keep track of the externalities χ_{N1}, χ_{N2} and prices of housing services ρ_{N1}, ρ_{N2} in both neighborhoods, for the sake of exposition we refer to this vector as

$$\theta \equiv (\chi_{N1}, \chi_{N2}, \rho_{N1}, \rho_{N2}).$$

We again start by analyzing a model in which the externalities χ_{N1}, χ_{N2} and prices of housing services ρ_{N1}, ρ_{N2} are not equilibrium objects, but rather are externally set to some fixed values (recall that the wage ω is also partial equilibrium in our model).

Again define $\mathcal{S}$ to be the product $\sigma-$algebra generated by $\mathcal{B}(\mathbb{H}_\theta)$, $\sigma\big(\{a_i\}\big)$, and $\sigma\big(\{n\}\big)$ containing subsets of the form $B = \mathcal{H} \times \mathcal{A} \times \mathcal{N}$. Now there is a residential decision rule $g_{n;\theta} : \mathbb{H}_\theta \times \mathbb{A} \times \{N1, N2\} \to \{N1, N2\}$ in addition to the human capital decision rule $g_{h;\theta} : \mathbb{H}_\theta \times \mathbb{A} \times \{N1, N2\} \to \mathbb{H}_\theta$. Note that because there are no moving costs and externalities/prices are set outside the model, the state space and domain of the decision rules could also be restricted to $\mathbb{H}_\theta$ and $\mathbb{A}$. Given these measurable decision rules and the Markov chain ability process alternatively written using the function $Q : \mathbb{A} \times \sigma\big(\{a_i\}\big) \to [0,1]$ defined by

$$Q\left(a_i, \mathcal{A}\right) \equiv \sum_{a_j \in \mathcal{A}} \pi(a_j | a_i),$$

we know from Theorem 9.13 in SLP that we can write a well-defined transition function P_θ : $S_\theta \times \mathcal{S}_\theta \to [0,1]$ for the Markov process induced by the household's problem as:

$$P_\theta\Big((h, a_i, n_-),\ \mathcal{H}\times\mathcal{A}\times\mathcal{N}\Big) \ = \ \mathbf{1}\Big\{g_{n;\theta}\left(h, a_i, n_-\right) \in \mathcal{N}\Big\}\ \mathbf{1}\Big\{g_{h;\theta}\left(h, a_i, g_{n;\theta}\left(h, a_i, n_-\right)\right) \in \mathcal{H}\Big\}\ Q\left(a_i, \mathcal{A}\right) \tag{16}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. The Markov process described by P_θ induces a mapping

$$\Psi_\theta : \mathcal{P}(\mathcal{S}_\theta) \to \mathcal{P}(\mathcal{S}_\theta)$$

from the set of probability measures $\mathcal{P}(\mathcal{S}_\theta)$ into itself that updates probability measures as follows:

$$\widehat{\mu}_{i+1,\theta}(B) = \Psi_\theta(\widehat{\mu}_{i,\theta}(B)) = \int P_\theta\big(x, B\big)\ \widehat{\mu}_{i,\theta}(dx)$$

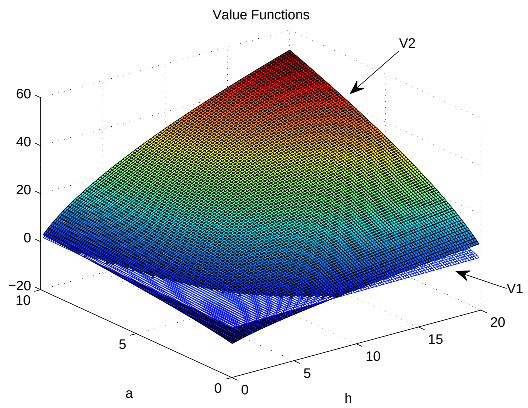
A proof would require the definition of a subset of the parameter space

$$\Theta^\star = \Big\{(\chi_{N1}, \chi_{N2}, \rho_{N1}, \rho_{N2}) \in \mathbb{H}_\theta\times\mathbb{H}_\theta\times\mathbb{P}\times\mathbb{P} :\ \ \chi_{N1} \leq \chi_{N2},\ \ \rho_{N1} \leq \rho_{N2}, \quad \text{and} \quad \phi(\chi_{N1}, \chi_{N2}, \rho_{N1}, \rho_{N2}) \leq 0\Big\}$$

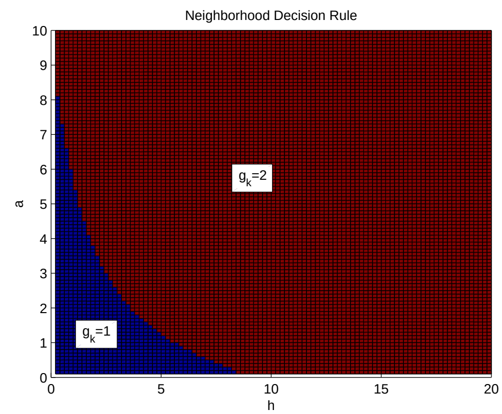
where the last condition would ensure that under a parameterization $\theta_i \in \Theta^\star$, the parameter θ_{i+1} (externalities and prices) implied by the model is itself in $\Theta^\star$.

It is difficult to analytically determine what restrictions ϕ would guarantee that $\Theta^\star$ is a compact, convex set for which the parameters implied by the model under a parameterization in $\Theta^\star$ is a self-map. We generally would need the following to hold: At low levels of h and a, the lower price of housing in N1 is worth more to a household than is the higher externality in N2. This would ensure that the implied externality in N1 would be less than the implied externality in N2. At the same time, this could not be true for too many households, or else enough people would want to live in N1 so that the implied (by the optimizing decision rules and current distributions) price of housing there would be higher than in N2.

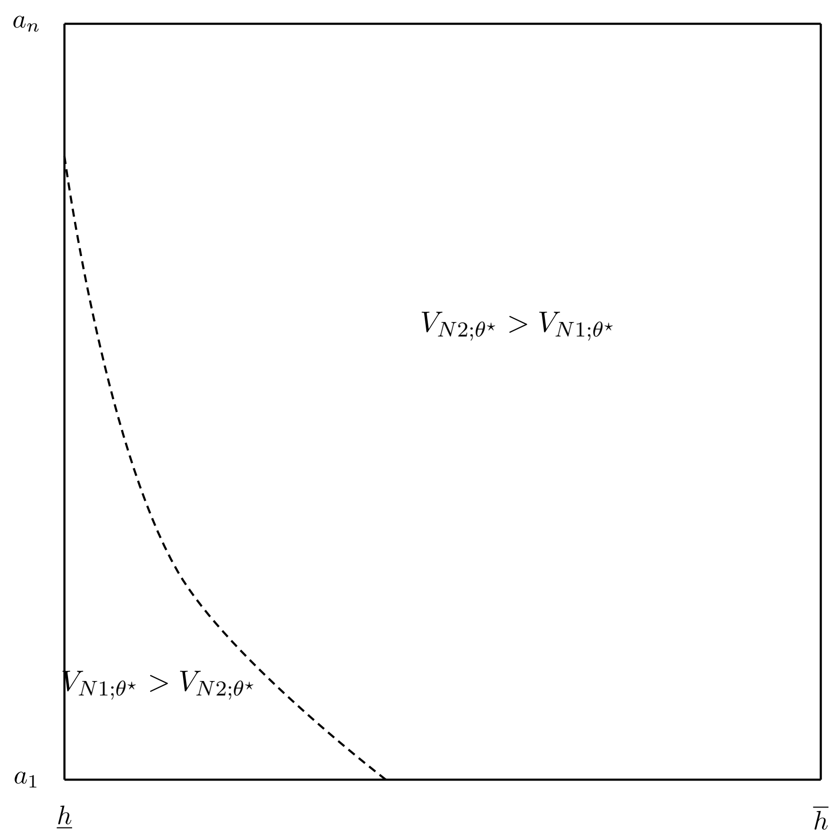
Figure 2 shows two example value functions satisfying these conditions. Here it is worth pointing out a computational difficulty, which is that the neighborhood-specific value functions V_{N1}, V_{N2} : $\mathbb{H}_\theta \times \mathbb{A} \to \mathbb{R}$ are not simply the value functions of each neighborhood in the model under segregation (ie, the value functions $V_{(\chi_{N1},\rho_{N1})}$ and $V_{(\chi_{N2},\rho_{N2})}$ of the two associated SRCEs with prices set outside the models). These value functions are the value of counterfactually residing in either neighborhood this period, with a continuation value that also accounts for future mobility between neighborhoods (ie, the value functions $V_{N1;(\chi_{N1},\chi_{N2},\rho_{N1},\rho_{N2})}$ and $V_{N2;(\chi_{N1},\chi_{N2},\rho_{N1},\rho_{N2})}$ with partial equilibrium prices) depending on where in $\mathbb{H}_\theta \times \mathbb{A}$ the household moves.



(a) Value Function Manifolds for $(\chi_{N1}, \chi_{N2}, \rho_{N1}, \rho_{N2})$



(b) Nbd Decision Rule for $(\chi_{N1}, \chi_{N2}, \rho_{N1}, \rho_{N2})$



(c) The Projection of $V_{N1;\theta^\star} \cap V_{N2;\theta^\star}$ onto $\mathbb{H} \times \mathbb{A}$ for a fixed $\theta^\star \in \Theta^\star$

Figure 2: Value Functions for Parameters $(\chi_{N1}, \chi_{N2}, \rho_{N1}, \rho_{N2})$ Likely to Be in $\Theta^\star$

Again, we have not analytically specified a rule ϕ defining $\Theta^\star$. However, if we could do so, we could show that the model with moving also satisfies the hypotheses of Theorem 2 in Hopenhayn and Prescott (1992), which would serve as a proof that there is a unique stationary distribution associated with each $\theta^\star$.

Given the unique invariant distribution $\widehat{\mu}^*_{\theta^\star}$, we would re-define

$$J : \mathbb{H}^\star \times \mathbb{H}^\star \times \mathbb{P}^\star \times \mathbb{P}^\star \to \mathbb{H}^\star \times \mathbb{H}^\star \times \mathbb{P}^\star \times \mathbb{P}^\star,$$

or

$$J : \Theta^\star \to \Theta^\star,$$

by:

$$\begin{aligned} J_1(\theta^\star) = \chi_{N1}(\theta^\star) &= \frac{\int h\, \mathbf{1}\big\{g_{n;\theta}(h,a,n_-) = N1\big\}\, d\widehat{\mu}^*_{\theta^\star}}{\int \mathbf{1}\big\{g_{n;\theta}(h,a,n_-) = N1\big\}\, d\widehat{\mu}^*_{\theta^\star}} \\ &= \frac{\int h\, d\mu^*_{\theta^\star}}{\int d\mu^*_{\theta^\star}} \end{aligned}$$

$$J_2(\theta^\star) = \chi_{N2}(\theta^\star) = \frac{\int h\, d\mu^*_{\theta^\star}}{\int d\mu^*_{\theta^\star}}$$

$$J_3(\theta^\star) = \rho_{N1}(\theta^\star) = f\left(\int g_{s;\theta^\star}\left(h,a,g_{n;\theta^\star}(h,a,n_-)\right)\, \mathbf{1}\Big\{g_{n;\theta}(h,a,n_-) = N1\Big\}\, d\widehat{\mu}^*_{\theta^\star} \right)$$

$$J_4(\theta^\star) = \rho_{N2}(\theta^\star) = f\left(\int g_{s;\theta^\star}\left(h,a,g_{n;\theta^\star}(h,a,n_-)\right)\, \mathbf{1}\Big\{g_{n;\theta}(h,a,n_-) = N2\Big\}\, d\widehat{\mu}^*_{\theta^\star} \right)$$

By the definition of $\Theta^\star$, we would know that

$$J : \Theta^\star \to \Theta^\star$$

is a self-map. Thus we would again appeal to Theorem 12.13 of SLP as proof that J is continuous, and then again apply Brouwer's Fixed Point Theorem to show existence of a general equilibrium.

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